Evolutionary Implementation with Estimation of Unknown Externalities

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Abstract

We consider an implementation problem in congestion games with players having heterogeneous costs of taking actions. The planner wishes to design a price scheme under which congestion externalities are internalized in the long run regardless of the initial states. However, he does not fully know the players’ cost functions and can observe only the aggregate strategy distribution, which compels him to estimate the externalities. We present an estimation procedure such that imposing the estimates of externalities on players over time makes an $\epsilon$-optimum globally stable under the best response dynamics for sufficiently small payoff noises if externalities among players taking the same action are sufficiently larger than those among players taking different actions.

JEL classification: C72, C73, D62, H41.

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1 Introduction

In this paper, we consider an implementation problem in the class of congestion games introduced by Rosenthal (1973). In our game, each player chooses an action from a finite choice set, and taking the action has a cost. The game is called a congestion game because the cost of taking the action increases with the total number of players who take the same action. Congestion causes equilibrium to be generally inefficient. The planner would like to achieve an efficient state but lacks thorough knowledge of the players’ cost functions.

We are interested in congestion problems in large economies, such as congestions in road traffic and Internet connections, and common-pool problems like international fisheries. Therefore, we consider a nonatomic game with a continuum of anonymous players.\(^1\) In general, to achieve an optimum, we could consider a traditional one-shot revelation mechanism such as the Vickrey-Clarke-Groves (VCG) mechanism. However, such a revelation mechanism is labor-intensive for the planner because he must collect reports from the agents and compute allocations for each set of reports. Further, because our game’s planner allocates actions to anonymous players, he must write forcing contracts to ensure that players obey his directions.\(^2\) Therefore, the mechanism is difficult to carry out, particularly in large economies such as our model. Moreover, we can show by example that a dominant strategy implementation may be impossible if the agents are anonymous.\(^3\) Therefore, the next step would be a Nash implementation. However, because Nash equilibrium requires each agent to correctly anticipate other agents’ actions, a realistic situation, particularly in large economies, is that the agents gradually

\(^1\)If players are anonymous, their allocation does not depend on their identities.
\(^2\)In the case of road traffic congestion, the planner has to ensure that the players use the routes that he assigns.
\(^3\)See Sandholm (2005) for an example.
learn to play equilibrium strategies. Therefore, it is important to consider a mechanism’s equilibrium outcome stability or investigate whether there exists an agents’ learning process that causes the outcome.

Sandholm (2002, 05) presents a novel framework to address the implementation problem in congestion games that is practical in large economies. In our congestion game, the planner must internalize the congestion externalities to achieve a social optimum. Sandholm (2002, 05) shows that if the planner can internalize the congestion externalities evaluated at the current state in each period, he can achieve a social optimum in the long run without collecting reports or writing forcing contracts.\footnote{Strictly speaking, we should say “at each instant of time” instead of “in each period” because Sandholm (2002, 05) (and our work) considers continuous-time models.} Specifically, Sandholm considers evolutionary Nash implementation via a price scheme in which, instead of collecting reports, the planner levies taxes on each action in each period, and the agents gradually learn to play equilibrium strategies according to an evolutionary dynamics. Under Sandholm’s price scheme, the planner only needs to internalize the congestion externalities evaluated at the realized state in each period. Therefore, this mechanism is not labor-intensive for the planner, and it allows agents to make mistakes in forecasting opponents’ strategies during learning periods. In addition, because evolutionary implementation requires the optimum to be globally stable under the evolutionary dynamics, agents learn to play socially optimal strategies from any initial state.

Theoretically, Sandholm’s result follows from the fact that the planner can create a potential game (Monderer and Shapley, 1996) by properly changing payoff functions through a price scheme. Once we obtain a potential game, we can describe the stability of the game’s equilibria by using an associated potential function because every local maximizer of the potential function is an asymptotically sta-
ble equilibrium under well-behaved evolutionary dynamics. Sandholm (2002, 05) demonstrates that if the planner makes players pay the values of congestion externalities evaluated at the current state, the resulting game is a potential game with the utilitarian social welfare function as its potential function.

This paper deals with the situation in which the planner cannot correctly compute the values of externalities even at the current state. To apply Sandholm’s price scheme, the planner must be able to internalize the externalities at the realized state, which requires that

1. the planner knows the players’ cost function;
2. players have homogeneous costs.

However, these assumptions can be restrictive in some applications. To see this, let us consider the example of traffic congestion in which agents choose whether to use a road. In this example, the cost of using the road is the time cost that is the travel time multiplied by the agents’ value of time (VOT). Thus, by assuming that the planner knows the cost function, we implicitly assume that he knows each agent’s VOT. In reality, though, it would be natural to believe that the VOT is private information.\(^5\) Moreover, the second assumption would not be desirable, particularly in large economies.\(^6\) To see why homogeneity is necessary for Sandholm’s scheme, suppose that the drivers are heterogeneous in terms of their VOT, valued as either high or low. Then, because the magnitude of externalities that the agents bear differs according to the VOT, the planner needs to know the number of drivers having high and low VOT to internalize the externalities at the current state.

\(^5\)By simulating the demand functions and travel cost function with data, Markose et al. (2007) compute optimal traffic under the assumption that the VOT is equal to an hourly wage. However, the VOT is generally different from the hourly wage, unless there is no distortion in the labor market. See, for example, Boardman et al. (2006) pp. 340-41.

\(^6\)Studying network models with heterogeneous agents is an active research subject in the operations research literature. See Perakis (2007) and references therein.
state. However, because the planner does not know the players’ type (i.e., VOT), he can observe only the total number of drivers. In other words, if the agents are heterogeneous, the planner needs information on the disaggregate state to compute the current values of externalities although he can observe only the aggregate state. Thus, the purpose of this paper is to weaken the two foregoing assumptions and achieve the evolutionary implementation of the social optimum when the cost function is heterogeneous and the planner does not know the cost functions.\(^7\)

Because the planner cannot compute the values of externalities resulting from his lack of knowledge of the cost functions and the heterogeneity of the players, it is no longer possible to guide the economy to a social optimum by internalizing the externalities at the current state in each period. However, because what the planner has to do is to internalize the externalities only in the long-run as in Sandholm (2002, 05), it is not necessary that current externalities are always internalized on the way to the optimum. In this paper, we consider a framework in which, after observing the current state, the planner estimates the values of current externalities according to an estimation formula constructed in advance.\(^8\) The planner then imposes the estimated values of externalities on the agents as taxes in each period. In the context of road traffic congestion, the planner needs to estimate the weighted averages of VOTs for each action where the weight for a type is given by the type’s population taking the action, owing to the heterogeneity of the agents. The planner would like

\(^7\)On the other hand, we assume that the planner knows the players’ benefits from the actions. Although this might seem restrictive, we argue that, in the case of road traffic congestion, the benefit derived from going to a destination would be the wage if the trip purpose is commuting, and therefore it would be reasonable to assume that the benefits are observable to the planner. Contrary to us, Sandholm (2005) assumes that the planner knows the costs but not the benefits.

\(^8\)We do not consider any measurement error when the planner observes data. Therefore, unlike estimations in econometrics, there is no statistical element in our estimation. In view of this, using the term “estimation” could be confusing, but this term is standard in adaptive control theory that aims to guide trajectories of an unknown dynamical system to a target state when the planner observes data without measurement error. See, for example, Ioannou and Sun (1996).
to make estimates so that his taxes converge to the true values of externalities in the long run. Therefore, unlike Sandholm’s case, we need to explain how to estimate the values of externalities in constructing a price scheme.

Because the planner would like to implement the social optimum, it would be natural to use the optimality condition, which is also the equilibrium condition under Pigouvian tax policy, for estimation. Indeed, if the planner succeeds in imposing the true values of externalities upon the agents at equilibrium, the condition must be satisfied. Therefore, we take an estimation strategy whereby, in each period, the planner infers unknown parameters of the value of externalities for each action as if the optimality condition holds under his taxes at the current state. Under this estimation strategy, the planner acknowledges that his estimates are generally not correct because the agents do not play their equilibrium strategies. However, because the agents learn to do so in the long-run, the planner believes that his estimates eventually become correct and the optimality condition holds at equilibrium. Because our game’s planner cares about only the long-run outcome, he achieves his objective as long as the optimality condition holds in the long-run.

Under the above tax policy, the social optimum is indeed an equilibrium. However, owing to the heterogeneity of the agents, the social optimum depends on the population distribution over the Cartesian product of action and type sets, which we call a population state, although the tax policy can depend on only aggregate strategy distributions owing to the planner’s informational constraint. As a result, any population state can be an equilibrium, as long as it induces the same aggregate distribution as the one that the optimal population state (i.e., the social optimum) induces. Hence, there is generally a continuum of equilibria, and there is no hope to guarantee the attainment of the social optimum even if the planner succeeds in imposing the values of externalities evaluated at the optimum on players.
Thus, instead of seeking the exact implementation, we implement the social optimum approximately by introducing noise to the model in the form of payoff disturbance. We consider a class of perturbed games in which a Gumbel-distributed random variable is added to the payoffs, and we parameterize the games according to magnitude of the noise. The players take best responses actions subject to the random disturbances. The resulting *perturbed equilibrium* is given by a fixed point of the logit formula as a result of the distributional assumption. With the noise, we can avoid a continuum of equilibria. Moreover, if the planner takes the estimation strategy above, we expect that a perturbed equilibrium is close to the social optimum when the noise is sufficiently small, and we would like this equilibrium to be globally stable under a players’ learning process. This leads us to consider the following implementation concept. That is, we say that a price scheme achieves the evolutionary implementation of the social optimum if, for any $\varepsilon > 0$, there is a threshold value of the Gumbel-distribution’s variance such that any perturbed equilibrium is $\varepsilon$-close to the social optimum and the set of perturbed equilibria is globally stable, as long as the variance is smaller than the threshold value.

For an evolutionary dynamics that describes the agents’ learning process, we consider the *perturbed best response dynamics* (Hofbauer and Sandholm, 2007). To interpret people’s behaviors underpinning the dynamics, let us consider a stochastic adaptation process in which revision opportunities randomly arrive for agents over time, and agents who obtain them switch to their best response actions subject to the random disturbances. Therefore, the agents intend to take their (perturbed) best response actions but cannot always do so because of, for example, their cogni-

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9Our framework is closely related to the recently developing literature on the structural estimation of discrete games. However, econometricians usually assume that equilibrium is attained in every period. If the players do not necessarily play equilibrium as in our model, constructing an objective function such as the likelihood function and obtaining consistent estimates would not be straightforward. See Ellickson and Misra (2011) for a survey of the literature.
tive constraints. If the population size is large, it follows that the perturbed best response dynamics closely approximates the process. Therefore, this dynamics is well-founded from a game-theoretic perspective.

For the global stability of equilibrium, one might want to use the properties of a potential game following Sandholm (2002, 05); however, if the cost function is heterogeneous, the properties no longer apply. Although potential games are characterized by externality symmetry such that “the marginal impact of new strategy $j$ users on current strategy $i$ users is the same as the marginal impact of new strategy $i$ users on current strategy $j$ users (Sandholm, 2005),” the heterogeneity of the cost function breaks this symmetry because marginal changes in the cost generally differ among people. Therefore, our game will never admit a potential function under any feasible price scheme. Thus, instead of the potential game approach, we invoke a sufficient condition for global stability provided by Hou (2005), who extend the Poincaré-Bendixson theorem, which holds in two-dimensional dynamical systems, to higher dimensions. This condition has a good economic interpretation that the externalities among players taking the same action are sufficiently large relative to those among players taking different actions.

The rest of this paper proceeds as follows. In Section 2, we state the basic structure of the model and define the social optimum. In Section 3, we define our implementation concept. In Section 4, we construct an estimation procedure and

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10 This type of adaptation process is commonly considered in the congestion game literature. See, for example, Friedman (1996) and Milchtaich (1996). They also study congestion games allowing heterogeneity in cost functions.

11 Our implementation concept is similar to the spirit of stochastic evolutionary implementation (Sandholm, 2007) that is defined with a stochastic evolutionary process under the same distribution as ours (i.e., a Gumbel distribution). The planner achieves the stochastic evolutionary implementation of the social optimum if the optimum is stochastically stable (Foster and Young, 1990): the invariant distribution of the stochastic evolutionary process puts probability one on the optimum as the noise vanishes. The result crucially depends on the fact that the game admits a potential function. Otherwise, it is generally difficult to characterize stochastically stable states.
demonstrate that the resulting price scheme achieves the evolutionary implementation of the social optimum. Section 5 is the conclusion and discusses subjects of future research. Proofs omitted from the main text are provided in an Appendix.

2 The Model

We consider a nonatomic game having a continuum of players with the common finite action set $A = \{0, 1, ..., K\}$. The whole population consists of $R$ populations. With the abuse of notation, $R$ also denotes the set of populations. Let $m^r$ be the mass of population $r \in R$. We normalize $\sum_{r \in R} m^r$, the whole mass of players, to one.

Let $z^r_k \in [0, m^r]$ be the mass of players in population $r$ who take action $k$. Then, $Z = \{z \in \mathbb{R}^{K+1}\times R : \forall r \in R, \sum_{k\in A} z^r_k = m^r\}$ is the set of player distributions over $A \times R$. We call $z \in Z$ a population state. A population state specifies the mass of players for each pair of action and type. Let $X = \{x \in \mathbb{R}^{K+1} : \exists z \in Z, \forall k \in A, x_k = \sum_{r \in R} z^r_k\}$. This is the set of player distributions over the action set and is obtained by aggregating population states over $R$ for each action (i.e., $x_k$ is the total mass of players taking action $k$). Hence, we call $x \in X$ an aggregate state.

We assume that payoffs are subject to random disturbances. Given $z \in Z$, the payoff from taking action $k$ for population $r$ is

$$U_k^r(z; \alpha, \theta, \eta) = \alpha_k^r - \theta^r T_k(x(z)) + \eta_k^r$$

for $k \in A_0$, $U_0^r(z; \alpha, \theta, \eta) = \eta_0^r$ \hspace{1cm} (1)

where $A_0 = A \setminus \{0\}$, $\alpha = \{\alpha_k^r\}_{(k,r)\in (A_0\times R)}$, $\theta = \{\theta^r\}_{r\in R}$, $\eta = \{\eta_k^r\}_{(k,r)\in A\times R}$, and $x(z)$ is the aggregate state induced by $z$ (i.e., $x_k(z) = \sum_{r \in R} z^r_k$ for each $k \in A$). While $\alpha$ and $\theta$ are fixed, we assume that each random disturbance $\eta_k^r$ is i.i.d. distributed according to the distribution function $F$. For $k \in A_0$, $\alpha_k^r + \eta_k^r$ is the random benefit from taking
action $k$ for population $r$. If the mean of $\eta^r_k$ is zero, we can interpret $\alpha^r_k$ as the mean benefit from action $k$ in population $r$. On the other hand, $\theta^r T_k(x)$ is the cost of taking action $k$ for population $r$ where $\theta^r$ is the cost parameter of population $r$ and $T_k(x)$ is a twice continuously differentiable function that depends only on aggregate states. We assume that the Jacobian of $T(x) = (T_k(x))_{k \in A_0}$ is positive definite for all $x \in X$. In particular, $\partial T_k(x)/\partial x_k > 0$ for all $k \in A_0$ so that congestion externalities arise.

Finally, the payoffs from action $0$ are given by only random disturbances. Thus, action $0$ is interpreted as the outside option.

We present examples below that fit with our framework.

**Example 1 (Traffic Congestion).** The economy consists of the set $\Psi$ of streets with typical element $\psi$. Players choose a route that is a collection of streets $[\psi] \subseteq \Psi$. The action set is then the set of available routes. (action $0$ is the action of not traveling). Let $x_i$ be the total mass of players choosing route $i$. The total traffic on street $\psi$ is then $y_\psi(x) = \sum_{i \in p(\psi)} x_i$ where $p(\psi) \subseteq A_0$ is the set of routes that contain street $\psi$. Thus, $y = (y_\psi)_{\psi \in \Psi}$ can be written as $y = \Omega x$ for some matrix $\Omega$ and $x \in A_0$. This matrix is called the link-path incidence matrix in the transportation science. In this example, what the planner can directly observe is the traffic distribution over streets, not the aggregate state that is the traffic distribution over routes. Thus, we assume $\Omega$ is non-singular so that the planner can figure out $x$ from $y$ as $x = \Omega^{-1} y$.

Let $t_\psi(y_\psi(x))$ be the travel time to go along street $\psi$. The travel time of route $i$ is then $T_i(x) = \sum_{\psi \in q(i)} t_\psi(y_\psi(x))$ where $q(i) \subseteq \Psi$ is the set of streets that comprises route $i$. Let us interpret $\theta^r$ as the value of time of population $r$. The travel cost of route $i$ for population $r$ is then $\theta^r T_i(x)$. 

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In Figure 1, there are one origin denoted by the black circle and two destinations denoted by the white circles. The set of streets is $\Psi = \{A, B, C\}$ and the set of routes is $A_0 = \{1, 2, 3\}$ where $q(1) = \{A\}, q(2) = \{B, C\}$, and $q(3) = \{B\}$. Thus, in this network, $\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This matrix is non-singular.

**Example 2 (Fishery).** This example is related to "fish wars," or more generally, "tragedy of the commons."\(^{12}\) There are a unit mass of fishermen. They catch one unit of fish in a fishery ground. The set of fishermen is divided into $R$ populations according to technology level of fishing. There are $K$ species of fish, and players choose which species of fish to catch (action 0 is the action of not fishing). For example, some fishermen specialize in catching tuna while some other fishermen specialize in catching squid. $x_i$ is then the mass of players catching species $i$. We interpret $\theta^r T_i(x)$ as the cost of fishing species $i$ for population $r$. Naturally, $T_i(x)$ is increasing in $x_i$ because catching a fish requires greater efforts as the mass of players catching the same species increases. On the other hand, we allow $\partial T_i(x)/\partial x_j \leq 0$ for $i \neq j$. This may be possible due to factors related to a food chain, which are called "biological externalities (Fischer and Mirman, 1996)." $\alpha^r_k$ is the (mean) revenue from selling a unit of species $k$ in the market of population $r$. Each population can have its own market. For example, we may interpret populations as countries.

At equilibrium, players take best response actions subject to the random disturbances. That is, given $\eta_r$, a positive mass of population $r$ takes action $k$ when

\(^{12}\)Huang and Smith (2014) estimate a discrete-choice game of fishing. See also Chiarella et al. (1984) who present a formal game-theoretic model of fishing.
\[ k \in \arg \max_{\ell \in A} U'_\ell(z; \alpha, \theta, \eta). \]

Let
\[
U'_k(x; \alpha, \theta) = \alpha'_k - \theta' T_k(x) \quad \text{for } k \in A_0, \\
U'_0(x; \alpha, \theta) = 0.
\]

These are the systematic payoffs that players receive when there is no random disturbance. The choice probability of action \( k \) in population \( r \) is then
\[
\Phi'_k(z) = \int_{\eta'} 1(\eta'_\ell - \eta'_k < U'_k(x(z); \alpha, \theta) - U'_\ell(x(z); \alpha, \theta) \forall \ell \neq k) \, dF(\eta')
\]
where \( 1(\cdot) \) is the indicator function and, with abuse of notations, \( F(\eta') = \prod_{\ell=0}^K F(\eta'_\ell) \).

As in Hofbauer and Sandholm (2007), we call \( \tilde{B}'_k(z) = \Phi'_k(z)m' \) the perturbed best response function. Assuming that the law of large numbers holds, \( \tilde{B}'_k(z) \) is the mass of players taking action \( k \) in population \( r \) when each player plays a best response to \( z \). We then look at a perturbed equilibrium that is defined by a fixed point of \( \tilde{B} \) (i.e., a population state \( \tilde{z} \) such that \( \tilde{z}'_k = \tilde{B}'_k(\tilde{z}) \) for all \((k, r) \in A \times R\)). The perturbed equilibrium is closely related to the quantal response equilibrium (McKelvey and Palfrey, 1995) for finite-player normal-form games.

In what follows, we assume that the random disturbances, which are i.i.d. distributed, have an extreme value distribution:
\[
F(\eta'_k) = F^\lambda(\eta'_k) \equiv \exp\left(-\exp(-\lambda \eta'_k)\right),
\]
where \( \lambda \in [0, \infty) \) is called the dispersion parameter. The distributional assumption
above yields the logit model in which a perturbed equilibrium $z$ satisfies

$$z_r^k = \frac{\exp[\lambda u^*_r(x(z); \alpha, \theta)]}{\sum_{\ell \in A} \exp[\lambda u^*_\ell(x(z); \alpha, \theta)]} m^r$$

for all $(k, r) \in A \times R$, (6)

In view of this, we call our game a *logit game*. We parameterize our game by $\lambda$ and denote it by $G^\lambda(\alpha, \theta) = \langle \{u^*_r(\cdot; \alpha, \theta)\}_{(k, r) \in A \times R}, F^A \rangle$. Because the variance of $\eta^*_k$ is $\pi^2/(6\lambda^2)$, the random disturbance vanishes as $\lambda \to \infty$. Given a sequence of logit games $\{G^\lambda(\alpha, \theta)\}_{\lambda}$, we denote the limiting unperturbed game by $G(\alpha, \theta) = \lim_{\lambda \to \infty} G^\lambda(\alpha, \theta)$. By (6), $z^*_r = \frac{m^r}{k+1}$ if $\lambda = 0$ whereas, as $\lambda \to \infty$, $z$ satisfies

$$\forall r \in R, \forall k \in A, z^*_r > 0 \Rightarrow k \in \arg\max_{\ell \in A} u^*_r(x(z); \alpha, \theta).$$

That is, if $\bar{z}$ has a positive mass of players who take action $k$ in population $r$, then action $k$ is a (unperturbed) best response to $\bar{z}$ for population $r$. This exactly means that $\bar{z}$ is a Nash equilibrium of $G(\alpha, \theta)$.

### 2.1 Social Welfare

As we will see, our goal is (approximate) implementation of the limiting unperturbed game’s optimum as in Tumennasan (2013) who considers revelation mechanisms for implementation in logit quantal response equilibria with vanishing noises. Thus, the social welfare is defined with respect to the deterministic payoff functions. In particular, we consider the utilitarian social welfare:

$$SW(z; \alpha, \theta) = \sum_{k \in A} \sum_{r \in R} z^*_r u^*_r(x(z); \alpha, \theta).$$

Given $(\alpha, \theta)$, let $z^* \in Z$ be the *social optimum* that maximizes $SW(z; \alpha, \theta)$ over $Z$. We assume that the social optimum is unique and interior. For $k \in A_0$, the
first-order condition of $z_k^*$ is

$$u'_k(x(z^*); \alpha, \theta) - \sum_{\ell \in A_0, r \in R} \left( \sum_{s \in R} z_s^r \theta_s \right) T_{\ell k}(x(z^*)) = 0 \quad (9)$$

where $T_{\ell k}(x) = \partial T_{\ell}(x)/\partial x_k$. Observe that dividing the optimality conditions by a constant does not alter the optimum. Thus, we may normalize $\theta^1$ to one. The value of externalities caused by action $k \in A_0$ is given by

$$\tau_k^*(z) = \sum_{\ell \in A_0} \left( \sum_{r \in R} z^r_{\ell} \theta^r \right) T_{\ell k}(x(z)). \quad (10)$$

Thus, condition (9) asks the planner to internalize the externalities.

As discussed in the introduction, we consider the case in which the planner does not have enough information to compute the values of externalities. Regarding the planner’s information, we make the assumptions summarized below.

**Assumption 1.**

1. The planner knows $\alpha$ and $T(\cdot)$ but not $\theta$.
2. The planner can observe aggregate states but not population states.

Because of conditions 1 and 2, the planner cannot compute the values of externalities even at the current state since, in view of (10), the values of externalities depend on population state $z$ and the cost parameter $\theta$.

To assess the plausibility of the informational assumptions above, let us revisit Example 1. If the purpose of players’ trip is commuting, $\alpha^r_k$ can be interpreted as the average wage of population $r$ in the destination of route $k$.\footnote{There is empirical evidence of locational wage variation for observationally equivalent workers.} Then, if population $r$...
refers to a players’ occupation, it might be reasonable to assume that \( \alpha_k' \) is observable to the planner. On the other hand, \( \theta_r' \) is interpreted as the value of time of players in population \( r \), which is generally unknown to the planner.

If the planner did not face any informational constraint, he could carry out the tax scheme \( \tau^* \) with \( \tau^* : Z \rightarrow \mathbb{R}^{K+1} \) in which \( \tau^*_k(z) \) is given by (10) for \( k \in A_0 \) and \( \tau^*_0(z) = 0 \). Denote the logit game under the tax scheme \( \tau^* \) by \( G^{\tau^*}(\alpha, \theta) = \langle \{u^*_k(\cdot; \alpha, \theta) - \tau^*_k(z)\}_{(k,r) \in A \times R}, F^1 \rangle \). In view of (7), a population state is an equilibrium of \( G^*(\alpha, \theta) = \lim_{\lambda \to \infty} G^{\tau^*}(\alpha, \theta) \) if and only if it is a KKT point of the problem \( \max_{z \in Z} SW(z; \alpha, \theta) \). Thus, \( G^*(\alpha, \theta) \) has the social optimum as its equilibrium. Moreover, if the social optimum is the unique KKT point, it is a unique equilibrium of \( G^*(\alpha, \theta) \), and thus the planner can achieve the social optimum with the scheme \( \tau^* \) whenever the agents play their equilibrium strategies. Unfortunately, though, this scheme is not feasible here because the planner cannot observe population states and does not know \( \theta \) by conditions 1 and 2 of Assumption 1. Thus, the planner needs to construct a price scheme that is feasible under his informational constraints to internalize the externalities in the long run.

3 Evolutionary Implementation

In this section, we define our implementation concept. As we argued in the introduction, our implementation problem is stated in a dynamic environment because we would like to guarantee that even if people behave adaptively rather than playing equilibrium strategies at once, the economy converges to optimum in the long run. Thus, we specify players’ adaptation process at first in Section 3.1 and then make the formal definition of implementation in Section 3.2.
3.1 Perturbed best response dynamics

In our implementation problem, we need to guarantee that the sets of perturbed equilibria of logit games under the planner’s policy is globally stable, but to do so, we need to specify a disequilibrium adaptation process of the players. In this paper, we consider the perturbed best response (PBR) dynamics (Hofbauer and Sandholm, 2007) that is defined by

\[ \dot{z} = \hat{B}(z) - z. \]  

Thus, the growth rate of \( z \) is given by the difference between a perturbed best response to \( z \) and \( z \) itself. We can provide a microfoundation for this dynamics as follows. Let us consider a stochastic adaptation process in a finite economy in which revision opportunities arrive for players according to an independent Poisson process, and players who obtain revision opportunities switch to their perturbed best response actions against the current state. Then, if the population size is large, this process is closely approximated by the PBR dynamics.

3.2 Price scheme

To formally state our implementation problem, we need to specify the type space on which the social choice function is defined. By Assumption 1, what the planner does not know about people’s preferences is \( \theta \). Because the planner knows \( \alpha \), our type space is then given by \( \Theta \subset \mathbb{R}_+^R \) that is the set of values \( \theta \) can take. We have already imposed some restrictions on the type space \( \Theta \). That is, given \( \alpha \), the social optimum is unique and interior for all \( \theta \in \Theta \). Then, given \( \alpha \), while a social choice function is given by map \( \phi : \Theta \to Z \) that specifies a population state for each
type, we particularly focus on the efficient social choice function

$$\phi^*(\theta) = \arg \max_{z \in Z} SW(z; \alpha, \theta).$$  \hfill (12)

By assumption, arg max_{z \in Z} SW(z; \alpha, \theta) is nonempty and singleton for all \( \theta \in \Theta \). Thus, this function is well-defined.

We define a price scheme \( \tau : X \to \mathbb{R}^{K+1} \) that specifies (lump-sum) tax levels for each action depending on aggregate states. We assume that the planner does not impose a tax on people who do not participate in the game (i.e., \( \tau_0(x) = 0 \)). This is reasonable because they do not cause any externality. Note that the domain of \( \tau \) is not \( Z \) because the planner can observe only aggregate states. The planner would like to implement \( \phi^* \) with a price scheme. The logit game played under a price scheme \( \tau \) is denoted by \( G^\lambda_{\tau}(\alpha, \theta) = \langle \{ u^\tau_k(\cdot; \alpha, \theta) - \tau_k \}_{(k, r) \in AXR}, F^1 \rangle \).

Let \( PE^\lambda_{\tau}(\alpha, \theta) \subseteq Z \) be the set of perturbed equilibria of logit game \( G^\lambda_{\tau}(\alpha, \theta) \). The planner constructs a price scheme so that all perturbed equilibria are located in a neighborhood of the social optimum. More specifically, for any \( \varepsilon > 0 \), the planner wishes to have \( PE^\lambda_{\tau}(\alpha, \theta) \subseteq N(z^*; \varepsilon) \) for sufficiently large \( \lambda \)'s where \( N(y; \varepsilon) \) is the open ball centered at \( y \in Z \) with radius \( \varepsilon \). Moreover, the planner aims to lead the economy to a perturbed equilibrium under his price scheme from any initial state \( z^0 \in Z \). Let \( \{ z(t) \}_{t \geq 0} \) be the trajectory of the PBR dynamics. We say that \( PE^\lambda_{\tau}(\alpha, \theta) \) is globally attracting under the PBR dynamics if \( \min_{z \in PE^\lambda_{\tau}(\alpha, \theta)} \| z(t) - z \| \to 0 \) as \( t \to \infty \) for any initial state \( z(0) = z^0 \in Z \). Then, we are now ready to define our implementation concept:

**Definition 1.** Given \( \alpha \), a price scheme \( \tau \) achieves the evolutionary implementation of a social choice function \( \phi \) if, for all \( \varepsilon > 0 \), there exists \( \bar{\lambda} \geq 0 \) such that, for all \( \lambda \geq \bar{\lambda} \), \( PE^\lambda_{\tau}(\alpha, \theta) \subseteq N(\phi(\theta); \varepsilon) \) and \( PE^\lambda_{\tau}(\alpha, \theta) \) is globally attracting under the perturbed best
response dynamics for all $\theta \in \Theta$.

Thus, if a price scheme $\tau$ achieves the evolutionary implementation of the efficient social choice function, any equilibrium of the logit game associated with the price scheme $\tau$ is almost optimal as long as $\lambda$ is sufficiently large, and the sets of equilibria are globally attracting under the perturbed best response dynamics for each of such $\lambda$’s. That is, as long as the degree of the random disturbance is sufficiently small, the planner can make agents play equilibrium strategies that are quite close to the optimal ones in the long run with the price scheme $\tau$.

4 Analysis

4.1 Estimation

As we have discussed, the planner cannot carry out Sandholm’s scheme because he does not have enough information to compute the values of externalities even at the current state. Therefore, we consider a scheme in which the planner estimates the values of externalities with the data available to him. Within our implementation framework, the planner cares about only the long-run outcome, so it is enough for him to make estimates so that they converge to the true ones in the long-run.\textsuperscript{15}

Let

$$\gamma_k = \sum_{r \in R} z_k^r \theta^r$$

for $k \in A_0$. This is the weighted average of the cost parameter where the weight for a population is given by the mass of agents taking action $k$ in the population. The

\textsuperscript{15}This argument is similar to the spirit of adaptive control that tries to guide the trajectory of an unknown dynamical system to a target state. See, for example, Ioannou and Sun (1996). However, whereas the target state is usually assumed to be known in adaptive control, it is unknown here.
value of externalities caused by action $k$ is then expressed as

$$\sum_{\ell \in A_0} \gamma_{\ell} T_{\ell k}(x).$$

(14)

Therefore, we let the planner estimate $\gamma = \{\gamma_k\}_{k \in A_0}$ to compute the estimated values of externalities. The planner’s estimator for $\gamma$ is a function $\hat{\gamma} : X \rightarrow \mathbb{R}^K$. Naturally, it is a function that depends only on observable aggregate states. At each instant of time, after observing aggregate state $x \in X$, the planner makes estimate $\hat{\gamma}(x)$, evaluates the values of externalities at the estimate by $\sum_{\ell \in A_0} \hat{\gamma}_{\ell}(x) T_{\ell k}(x)$, and imposes them on the players as taxes.

The problem is then how to estimate $\gamma$. Note that, if the planner succeeds in implementing the social optimum, the optimality condition (9) must hold at equilibrium when $\lambda \rightarrow \infty$. Therefore, we consider the following estimation strategy. That is, at each instant of time, the planner determines $\hat{\gamma}(x)$ so that the optimality condition holds at the current aggregate state $x \in X$ under his price scheme. Because the players behave according to the disequilibrium dynamics, the planner’s estimate is generally not correct (i.e., $\hat{\gamma}(x) \neq \gamma$). Thus, our price scheme does not completely internalize the externalities on the way to the target state unlike Sandholm’s scheme. However, the planner makes estimates so that the optimality condition holds at each instant of time in the belief that it actually holds when the agents play equilibrium strategies in the long run.

Specifically, the planner utilizes population 1’s optimality condition. Because we have normalized $\theta^1$ to one and assumed that the social optimum is interior, it is given by

$$\alpha^1_k - T_k(x^*) - \sum_{\ell \in A_0} \gamma^*_{\ell} T_{\ell k}(x^*) = 0 \quad \text{for all } k \in A_0,$$

(15)
where $\gamma^*_k = \sum_{r \in R} z^*_r \theta^r$. Given $x \in X$, the planner supposes that the above holds, and solves the resulting system of linear equations in $\gamma$. The system is written in matrix form as

$$
\begin{pmatrix}
T_{11}(x) & \cdots & T_{K1}(x) \\
\vdots & \ddots & \vdots \\
T_{1K}(x) & \cdots & T_{KK}(x)
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\vdots \\
\gamma_K
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_1^1 - T_1(x) \\
\vdots \\
\alpha_K^1 - T_K(x)
\end{pmatrix}.
$$

(16)

Because the Jacobian of $T$ is positive definite, (16) has a unique solution for all $x \in X$. Let the solution be $\hat{\gamma}(x)$. This serves as the planner’s estimate of $\gamma$ when the aggregate state is $x$. Although we assume that the social optimum is interior, it actually suffices for construction of $\hat{\gamma}(x)$ that the optimality condition binds for all actions for at least one population (i.e., $\exists r \in R, z^*_r > 0$ for all $k \in A$), as long as the planner is able to know for which population it is true.

Given $x \in X$, the planner’s taxes are the values of externalities estimated using $\hat{\gamma}(x)$. That is, we consider the price scheme $\hat{\tau}$ in which

$$
\hat{\tau}_k(x) = \sum_{\ell \in A_0} \hat{\gamma}_\ell(x) T_{\ell k}(x) \quad \text{for } k \in A_0
$$

(17)

and $\hat{\tau}_0(x) = 0$. Observe that this price scheme is feasible for the planner. We denote the logit game under the price scheme $\hat{\tau}$ by $\hat{G}^1(\alpha, \theta) = \langle \{u^*_k(\cdot; \alpha, \theta) - \hat{\tau}_k\}_{(k, r) \in A \times R}, F^1 \rangle$.

Because (16) has a unique solution for any $x \in X$, we particularly have

$$
\hat{\gamma}_k(x^*) = \sum_{r \in R} z^*_r \theta^r
$$

(18)

for all $k \in A_0$. Thus, the planner succeeds in imposing the true values of externalities on players at the efficient aggregate state when $\lambda \to \infty$. In view of this fact, one might think that the planner could implement the social optimum with the price
scheme \( \hat{\tau} \) in the limiting unperturbed game \( \hat{G}(\alpha, \theta) \equiv \lim_{\lambda \to \infty} \hat{G}^\lambda(\alpha, \theta) \). Indeed, the social optimum \( z^* \) is an equilibrium of \( \hat{G}(\alpha, \theta) \) because, by (18),

\[
\alpha_k^r - \theta^T_k(x^*) - \hat{\tau}_k(x^*) = 0 \quad \text{for all } (k, r) \in A_0 \times R. \tag{19}
\]

However, because (19) depends only on the aggregate state \( x^* \), we can see from (7) that any interior population state \( z \) that induces the same aggregate state as the social optimum induces (i.e., any \( z \in Z \) such that \( x(z) = x^* \)) is also an equilibrium of \( \hat{G}(\alpha, \theta) \). Hence, there is generally a continuum of equilibria. Observe that this is true for any price scheme, as long as it depends only on the aggregate state.

Thus, as long as the price scheme is feasible, there would be no hope to guarantee the global stability of the social optimum even if the planner succeeds in imposing the values of externalities evaluated at the social optimum on players. This actually motivates us to add noise to the model and consider the approximate implementation. In fact, because we can avoid a continuum of equilibria by considering logit games, we aim to find a sequence of perturbed equilibria under the price scheme \( \hat{\tau} \) that approaches the social optimum as \( \lambda \to \infty \). In the next section, we show that the PBR dynamics having a globally attractive rest point implies the existence of such a sequence under the price scheme \( \hat{\tau} \).

4.2 Global stability

For our implementation problem, we need to analyze the stability of the set of equilibria under the PBR dynamics (11). Before addressing the global stability, we seek a sequence of perturbed equilibria under the price scheme \( \hat{\tau} \) that converges to the social optimum as \( \lambda \to \infty \). After that, we impose a condition under which every trajectory of the PBR dynamics converges to a point on such a sequence for
each of sufficiently large $\lambda$’s. In general, though, the trajectory of the PBR dynamics does not necessarily attain a rest point owing to the possibilities of limit cycles or chaotic behaviors. However, the following lemma states that, if the trajectory of the PBR dynamics is convergent, we can construct a sequence of perturbed equilibria under the price scheme $\hat{\tau}$ that approaches the social optimum as $\lambda \to \infty$.

**Lemma 1.** Suppose that every trajectory of the PBR dynamics converges to a rest point under the price scheme $\hat{\tau}$. Then, there exists a sequence of perturbed equilibria under the price scheme $\hat{\tau}$ that converges to the social optimum $z^*$ as $\lambda \to \infty$.

**Proof.** See Appendix.

The outline of the above result’s proof proceeds as follows. At first, because the second-order optimality condition holds at the unique social optimum, it follows that a sequence of perturbed equilibria under the tax scheme $\tau^*$ exists that converges to the social optimum as $\lambda \to \infty$. Let $\{z^*(\lambda)\}$ be such a sequence. We then consider a difference equation that is a discretization of the PBR dynamics with time step $h$:

$$z^{i+1,h}(\lambda) = z^{i,h}(\lambda) + h \left( \hat{B}(x(z^{i,h}(\lambda))) - z^{i,h}(\lambda) \right)$$

(20)

where $z^{0,h}(\lambda) = z^*(\lambda)$, $h > 0$ is sufficiently small, and

$$\hat{B}_k(x) = \frac{\exp[\lambda(u'_k(x;\alpha,\theta) - \hat{\tau}_k(x))]}{\sum_{r \in A} \exp[\lambda(u'_r(x;\alpha,\theta) - \hat{\tau}_r(x))]} m^r \quad \text{for } (k, r) \in A \times R.$$ (21)

Let $z^h(\lambda)$ be an accumulation point of $\{z^{i,h}(\lambda)\}_{i \geq 0}$. Because $\hat{\tau}(x^*(\lambda)) \to \tau^*(z^*)$ as $\lambda \to \infty$ by (18) and $z^{0,h}(\lambda) = z^*(\lambda)$, an iterative argument on (20) implies $z^h(\lambda) \to z^*$ as $\lambda \to \infty$ for any small $h > 0$. However, because the difference equation (20) is a discrete-analogue of the PBR dynamics under the price scheme $\hat{\tau}$, it follows that its whole path converges to that of the PBR dynamics as $h \to 0$. Hence, if the trajectory
of the PBR dynamics starting at \( z^*(\lambda) \) converges to a rest point, it approaches the social optimum as \( \lambda \to \infty \). Let \( \hat{z}(\lambda) \) be such a rest point which is also a perturbed equilibrium. \( \{\hat{z}(\lambda)\} \) is then the sequence that we seek (See Figure 2).

\[
\dot{x}_k = \hat{B}_k(x) - x_k 
\]

for \( k \in A \) where \( \hat{B}_k(x) = \sum_{r \in R} \hat{B}_k^r(x) \). We call this dynamics the aggregate perturbed
best response (APBR) dynamics under the price scheme \( \hat{\tau} \). By the lemma below, although we seek a condition under which the PBR dynamics has a single-point global attractor, it suffices to identify a condition under which the APBR dynamics has a single-point global attractor for that purpose.

**Lemma 2.** If the APBR dynamics (22) has a single-point global attractor, then the PBR dynamics also has a single-point global attractor.

*Proof.* See Appendix.

To obtain our implementation result, we appeal to the condition of Hou (2005) that guarantees the existence of a single-point global attractor in high-dimensional dynamical systems.\(^{16}\) In view of Lemma 2, we apply the condition to the APBR dynamics. To begin with, we reproduce the result of Hou (2005) in the following.

**Theorem 1** (Hou, 2005). Consider a dynamics \( \dot{y} = f(y) \) where \( f : D \to \mathbb{R}^m \) is \( C^1 \) on \( D \subseteq \mathbb{R}^m \). Suppose that the dynamics has a unique trajectory for each \( y \in D \); \( D \) is positively invariant under the dynamics; and no trajectory approaches the boundary of \( D \) in a positive finite time. Then, if

\[
\max_k \frac{\partial f_k(y)}{\partial y_k} + \sum_{\ell \neq k} \left| \frac{\partial f_\ell(y)}{\partial y_k} \right| < 0
\]  

(23)

for all \( y \in D \), the dynamics has a single point global attractor.

Note that the existence of a rest point of the dynamics is not a condition but a result.\(^{17}\) It follows that the APBR dynamics, which is given by (22), fulfills all conditions in the theorem except for (23). In particular, the boundary is repelling under the APBR dynamics. Even if the initial point is on the boundary, the trajectory

\(^{16}\)Our technical approach for the stability analysis is different from the one taken by Sandholm (2002, 05). We elaborate on this in Section 4.3.

\(^{17}\)Moreover, Hou allows the dynamics to be non-\( C^1 \) on measure zero sets.
moves inward instantly and never returns to the boundary. For the APBR dynamics under the price scheme $\hat{\tau}$, (23) is written as

$$\max_{k \in A_0} \frac{\partial \hat{B}_k(x)}{\partial x_k} + \sum_{\ell \neq k} \left| \frac{\partial \hat{B}_\ell(x)}{\partial x_k} \right| < 1 \quad \text{for all } x \in X,$$

where $\hat{B}_k(x) = \sum_{r \in R} \hat{B}_r^k(x)$. We discuss an economic interpretation of the condition above after stating our implementation result.

We are now ready to state our main result. If there is $\bar{\lambda} < \infty$ such that (24) holds for all $\lambda \in [\bar{\lambda}, \infty)$, every trajectory of the APBR dynamics converges to a single global attractor for any $\lambda \geq \bar{\lambda}$. By Lemma 2, this is also true under the PBR dynamics. Thus, for any $\lambda \geq \bar{\lambda}$, every trajectory of the PBR dynamics converges to a single global attractor, and, by Lemma 1, this unique perturbed equilibrium must approach the social optimum as $\lambda \to \infty$. This exactly means that the planner succeeds in the evolutionary implementation of the social optimum.

**Proposition 1.** Suppose (24) holds for any large $\lambda$’s. Then, the price scheme $\hat{\tau}$ achieves the evolutionary implementation of the efficient social choice function $\phi^\ast$.

Our price scheme achieves an $\varepsilon$-optimum when the amount of noise in payoffs is sufficiently small, as long as (24) holds. In terms of dynamical systems, (24) is a high-dimensional analogue of Bendixson’s criterion that eliminates the possibility of limit cycles in planar dynamical systems, as discussed in Section 4.3. In the rest of this section, we discuss the interpretation of the condition in terms of economics.

Let $\hat{Q}_{kl}(x) = \frac{1}{\bar{\lambda}} \frac{\partial \hat{B}_l(x)}{\partial x_k}$. We then have

$$\hat{Q}_{kl}(x) = -\sum_{r \in R} \left( \frac{\partial \hat{C}_r^l(x)}{\partial x_k} - \sum_{n \in A_0} \frac{\partial \hat{C}_r^l(x)}{\partial x_n} \hat{B}_n^r(x) \right) \hat{B}_k^r(x) m^r,$$

where $\hat{B}_k(x) = \sum_{r \in R} \hat{B}_k^r(x)$. We discuss an economic interpretation of the condition above after stating our implementation result.
where \( \hat{C}_r(x) = \theta^r T_k(x) + \hat{\tau}_k(x) \). Then, (24) is rewritten as

\[
\max_{k \in A_0} \hat{Q}_{kk}(x) + \sum_{\ell \neq k} |\hat{Q}_{\ell k}(x)| < 1/\lambda \quad \text{for all } x \in X. \tag{26}
\]

In view of (25), \( \hat{Q}_{kk}(x) \) is the weighted sum of

- "The externality of action \( \ell \) on action \( k \) \( \left( \frac{\partial \hat{C}_r(x)}{\partial x_\ell} \right) \)"
- "The (weighted) average externality caused by action \( \ell \) \( \left( \sum_{n \in A_0} \frac{\partial \hat{C}_r(x)}{\partial x_\ell} B_r(x) \right) \)"

over the populations. In other words, it represents the magnitude of the externality of action \( \ell \) on action \( k \) relative to the average externality caused by action \( \ell \). Hence, although action \( \ell \) generally causes externalities to all actions, its externality on action \( k \) is particularly large if \( |\hat{Q}_{\ell k}(x)| \) is large. Note that, for condition (26) to be satisfied for large \( \lambda \)'s, \( \hat{Q}_{kk}(x) \) should be sufficiently negative because all the other terms are positive and \( 1/\lambda \) is close to zero when \( \lambda \) is large. However, because it represents the relative magnitude of actions’ own externalities (say, the externalities among players taking the same action) whereas the other terms represent the relative magnitudes of cross externalities (say, the externalities among players taking different actions), the condition requires that the actions’ own externalities are sufficiently large relative to the cross externalities.

In the example of road traffic congestion, the own externalities are caused by players using the same route, whereas the cross externalities are caused by players using different routes. To illustrate how the condition above works for stability, let us consider a case in which there are three routes \( a, b, \) and \( c \) as in Figure 3. Suppose that some players switch to route \( a \) from routes \( b \) and \( c \). The cost of route \( a \) then increases because route \( a \) has more traffic, whereas the cost of route \( c \) decreases.
because route $c$ has less traffic. To consider the change in the cost of route $b$, we divide the route into streets 1 and 2. Then, because some players switch from route $b$ to route $a$, the cost of street 2 decreases owing to route $b$’s own externalities. On the other hand, because some players switch from route $c$ to route $a$ and, as a result, traffic on street 1 increases, the cost of street 1 increases owing to cross externalities. Thus, if the own externalities are larger than the cross externalities, the cost of route $b$ decreases. In this case, the cost of route $a$ increases whereas the costs of routes $b$ and $c$ decrease, which would push the economy back toward the original state. On the other hand, if the cross externalities are larger than the own externalities, the cost of route $b$ increases. Therefore, every player wants to switch to route $c$, which would push the economy further away from the original state.

![Figure 3: Three routes](image)

### 4.3 Potential Functions

Sandholm (2002, 05) considers unperturbed games and uses the fact that the game under his price scheme has a potential function for the stability analysis. We say that a game admits a **potential function** if its partial derivative with respect to $z_k^r$ is the payoff of the game from action $k$ for population $r$ for all $(k, r) \in A \times R$. A game that has a potential function is called a **potential game**, which is originally introduced by Monderer and Shapley (1996). If a potential function exists, every equilibrium that locally maximizes the function is locally stable under well-behaved
evolutionary dynamics (see Sandholm, 2001). Then, Sandholm (2002, 05) shows that if the planner internalizes the externalities evaluated at the current aggregate state at each instant of time, the resulting game has the social welfare function as its potential function. Assuming that the social welfare function is strictly concave, it then follows that the optimum is globally stable under the evolutionary dynamics.

Unlike Sandholm (2002, 05), we consider perturbed games, but it follows that, if the limiting unperturbed game has a potential function, the stability of the PBR dynamic’s rest points, and hence, perturbed equilibria, could be analyzed with the potential function by Theorem 3.2 of Hofbauer and Sandholm (2007). However, the following proposition shows that it is never possible for us to construct a potential function owing to the heterogeneity of the cost functions.

**Proposition 2.** Suppose there exists \( z \in Z \) such that the payoff is continuously differentiable at \( z \). If the cost function is heterogeneous, then \( G_\tau = \{ u_k'(x_k) - \tau_k \}_{(k,r) \in A \times R} \) is not a potential game for any feasible price scheme \( \tau \) that depends on only aggregate states.

**Proof.** Let \( |R| \geq 2 \). Suppose \( G_\tau \) is a potential game and the payoff vector is continuously differentiable at \( z \). Then, for all \( k, \ell \in A \) and \( s, r \in R \), we obtain

\[
\frac{\partial (u_k'(x(z)) - \tau_k)}{\partial z^s_k} = \frac{\partial (u_\ell'(x(z)) - \tau_\ell)}{\partial z^r_\ell} \tag{27}
\]

by taking the cross derivatives of a potential function for \( G_\tau \) with respect to \( z^s_k \) and \( z^r_\ell \). Therefore, for \( k = \ell \) and \( r \neq s \),

\[
-\theta' \frac{\partial T_k}{\partial x_k} - \frac{\partial \tau_k}{\partial z^s_k} = -\theta' \frac{\partial T_k}{\partial x_k} - \frac{\partial \tau_k}{\partial z^r_\ell} \Rightarrow \frac{\partial \tau_k}{\partial z^s_k} = \frac{\partial \tau_k}{\partial z^r_\ell} = (\theta' - \theta') \frac{\partial T_k}{\partial x_k} \Rightarrow \frac{\partial \tau_k}{\partial z^s_k} \neq \frac{\partial \tau_k}{\partial z^r_\ell}.
\]

Then, \( \tau \) is not feasible because if it is, \( \tau_k \) can depend on \( z^s_k \) and \( z^r_\ell \) only through \( x_k \) and hence, \( \partial \tau_k / \partial z^s_k = \partial \tau_k / \partial z^r_\ell = \partial \tau_k / \partial x_k \).

\[\square\]
Therefore, as long as we consider standard payoff functions, the analytical technique that Sandholm (2002, 05) uses is not applicable here. Note that equation (27) says externalities are symmetric. That is, player \(i\)'s externality on player \(j\) is the same as player \(j\)'s externality on player \(i\). Because the externalities caused by a player in our model are captured by his effects on the costs of other players, the symmetry is broken if the cost function is heterogeneous, whereas it is preserved if the cost function is homogeneous. This illustrates that it would be more difficult to guarantee the stability of equilibria under heterogeneous economies.

Recall that, if the trajectory of the PBR dynamics attains a rest point, we can find a sequence of perturbed equilibria under the price scheme \(\hat{\tau}\) that approaches the social optimum as \(\lambda \to \infty\) by Lemma 1. Therefore, we want to guarantee that every trajectory converges to the same rest point. Although we know that any trajectory is bounded because \(Z\) is positively invariant under the PBR dynamics, we still need to eliminate the possibility of limit cycles.\(^{18}\)

There are two main approaches that can be taken. The first one is to find a Lyapunov function for the dynamics. If a Lyapunov function exists, no trajectory can cycle because every trajectory ascends the function and stops at a local maximum. In particular, the potential function is one particular Lyapunov function. Thus, even if a potential function does not exist, it may still be possible to find a Lyapunov function, but unfortunately, there is generally no systematic way to find one. The second approach is to use the Poincaré-Bendixson theorem, though the original theorem is valid only for planar dynamical systems.\(^{19}\)

In view of the fact that it is not straightforward to find a Lyapunov function, we

\(^{18}\)Even if an equilibrium is unique and locally stable, it is not necessarily globally stable. Indeed, using a predator-prey model, Hofbauer and So (1990) provide an example in which a unique locally stable equilibrium is surrounded by limit cycles.

\(^{19}\)One can find the definition of the Lyapunov function and the statement of the Poincaré-Bendixson theorem in standard textbooks on dynamical systems. See, for example, Khalil (2001).
took the second approach. In fact, Hou’s (2005) theorem we invoked can be viewed as an extension of the Poincaré-Bendixson theorem to higher dimensions. Indeed, if (23) holds, \( \frac{\partial f_k(y)}{\partial y_k} < 0 \) on \( D \) for all \( k \), and thus \( \frac{\partial f_k(y)}{\partial y_k} + \frac{\partial f_\ell(y)}{\partial y_\ell} < 0 \) for all \( y \in D \). This is Bendixson’s criterion that eliminates the possibility of limit cycles in planar dynamical systems. Note, however, that, unlike planar systems, the nonexistence of a limit cycle does not necessarily imply that every trajectory converges to a rest point when there are more than two dimensions owing to the possibility of chaotic phenomena. This is the reason why Theorem 1 is not a simple extension of the Poincaré-Bendixson theorem.\(^{20}\)

5 Conclusion

We constructed a price scheme that achieves the evolutionary implementation of the efficient social choice function when people have heterogeneous costs and the costs are unknown to the planner, as opposed to Sandholm (2002, 05). Because the planner cannot calculate the values of externalities even at the current state in our environment, we constructed an estimation procedure for the values of externalities given the current aggregate state, and let players pay the estimated values of externalities over time. However, because a price scheme can depend on only aggregate states whereas players are heterogeneous, a continuum of equilibria arises even if the planner succeeds in imposing the true values of externalities on the players. Thus, we considered an approximate implementation by adding noise to the game in the form of payoff disturbance. Regarding the stability analysis, the planner is never able to construct a feasible price scheme such that the resulting game has a potential function owing to the heterogeneity of players. Therefore, we

\(^{20}\)For example, the Poincaré-Bendixson theorem does not require that the boundary be repelling under the dynamics.
invoked an extension of the Poincaré-Bendixson theorem to higher dimensions to guarantee the global stability of equilibrium.

However, there are several points that may limit the applicability of our scheme, which should be the subject of future research. First, because our planner cared about only the long-run outcome, we could focus on a particular mechanism. However, if the planner also cared about the speed of convergence to the optimum, we would have to consider the most effective mechanism among all possible ones. Second, although we focused on negative externalities, other sources of inefficiencies such as positive externalities and uncertainty are also important in real life.\(^{21}\) Third, whereas we assumed that people are myopic when learning their equilibrium strategies, the planner has to take the policy’s effects on people’s expectations into account if people are forward-looking. Last, we assumed that the people’s benefits from taking actions are known to the planner, although we found justification for the assumption in the road traffic application. Therefore, our final goal will be to construct a price scheme that achieves the evolutionary implementation of the social optimum when the planner knows neither the benefits nor the costs.

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\(^{21}\)Fujishima (2013) considers evolutionary implementation of optimal city size distributions when there are positive externalities known as agglomeration economies. Regarding uncertainty, a central bank usually wants to minimize the variabilities in economic variables such as inflation and real interest rates. There are several works that study schemes where, to make the target state stable, the central bank estimates the structural parameters of the economy when people gradually learn the economic environment. See, for example, Tesfaselassie et al. (2011).
Appendix

Proof of Lemma 1. Let us consider the maximization problem of the perturbed social welfare function

\[
SW^\lambda(z) \equiv SW(z) - \frac{1}{\lambda} \left( \sum_{k \in A_0} \sum_{r \in R} z_k^r \ln \frac{z_k^r}{m_r} + \left( m_r' - \sum_{\ell \in A_0} z_{\ell}^r \right) \ln \frac{m_r' - \sum_{\ell \in A_0} z_{\ell}^r}{m_r} \right). \tag{28}
\]

Note that a KKT point of \( \max_{z \in \Delta} SW^\lambda(z) \) is a perturbed equilibrium of \( G^\lambda \). Because the second-order sufficient condition holds at the unique social optimum \( z^* \), which is the maximizer of \( SW(z) \), we can take a sequence \( \{z^*(\lambda)\} \), where \( z^*(\lambda) \) is a maximizer of \( SW^\lambda(z) \), such that \( z^*(\lambda) \to z^* \) as \( \lambda \to \infty \).\(^2\) Observe that \( \{z^*(\lambda)\} \) is a sequence of perturbed equilibria under the tax scheme \( \tau^* \) converging to \( z^* \) as \( \lambda \to \infty \). Let \( x^*(\lambda) \) be the aggregate state induced by \( z^*(\lambda) \).

Consider the difference equation

\[
z^{i+1,h}(\lambda) = z^{i,h}(\lambda) + h(B(x^{i,h}(\lambda)) - z^{i,h}(\lambda)) \tag{29}
\]

where \( h > 0 \) is a sufficiently small real number, \( x^{i,h}(\lambda) \) is the aggregate state induced by \( z^{i,h}(\lambda) \), and \( B(x) \) is given in (21). We take \( z^*(\lambda) \) for \( z^{0,h}(\lambda) \). Let

\[
B_k^*(z) = \frac{\exp \left[ \lambda \left( u_k^r(x(z)) - \tau_k^r(z) \right) \right]}{\sum_{\ell \in A} \exp \left[ \lambda \left( u_{\ell}^r(x(z)) - \tau_{\ell}^r(z) \right) \right]} m_r' \quad \text{for } (k, r) \in A \times R, \tag{30}
\]

where \( \tau_k^r(z) = \sum_{\ell \in A_0} \sum_{r \in R} z_{\ell}^r \theta_{k}^r T_{\ell k}(x(z)) \) for \( k \in A_0 \) and \( \tau_0^r(z) = 0 \). Because \( \hat{x}(x^*) = \tau^*(z^*) \),

\(^{22}\)See Theorem 2G. 8 of Dontchev and Rockafellar (2009). In addition to the second-order sufficient condition of \( z^* \), we need the condition that the gradient of active constraints is linearly independent. Because our constraint is that \( z \) should belong to a simplex, this obviously holds.
\[ \dot{B}(x^*) = B^*(z^*) \]. Hence,

\[ \|z'^\lambda(\lambda) - \dot{B}(x^'(\lambda))\| = \|B^*(z'^\lambda(\lambda)) - \dot{B}(x^'(\lambda))\| \]

\[ \leq \|B^*(z'^\lambda(\lambda)) - B^*(z'^\lambda(\lambda))\| + \|\dot{B}(x^*) - \dot{B}(x'^\lambda(\lambda))\|. \] \hspace{1cm} (31)

By the mean value theorem, \( \dot{B}(x^*) - \dot{B}(x'^\lambda(\lambda)) = D\dot{B}(\bar{x}(\lambda))(x^* - x'^\lambda(\lambda)) \) where \( D\dot{B} \) is the Jacobian of \( \dot{B} \) and \( \bar{x}(\lambda) \) is an aggregate state on the line segment connecting \( x^* \) and \( x'^\lambda(\lambda) \). Because \( \sum_{(k,r)\in A\times R} \partial B_k^\ell(x)/\partial x_\ell = 0 \) for any \( \ell \in A \), \( D\dot{B}(\bar{x}(\lambda)) \) is bounded for any \( \lambda \). Because the same argument applies to \( B^*(z'^\lambda(\lambda)) - B^*(z'^\lambda(\lambda)) \), (32) goes to zero as \( \lambda \to \infty \). Hence, \( \|z^{0,h}(\lambda) - z^{1,h}(\lambda)\| \to 0 \) as \( \lambda \to \infty \). But then, we have

\[ \|\dot{B}(x^{1,h}(\lambda)) - z^{1,h}(\lambda)\| \]

\[ \leq \|\dot{B}(x^{1,h}(\lambda)) - \dot{B}(x^{0,h}(\lambda))\| + \|\dot{B}(x^{0,h}(\lambda)) - z^{0,h}(\lambda)\| + \|z^{0,h}(\lambda) - z^{1,h}(\lambda)\| \to 0 \]

and hence \( \|z^{2,h}(\lambda) - z^{1,h}(\lambda)\| \to 0 \) as \( \lambda \to \infty \). Therefore, we can induce \( \|z^{i+1,h}(\lambda) - z^{i,h}(\lambda)\| \to 0 \) as \( \lambda \to \infty \) and thus \( z^{i,h}(\lambda) \to z^* \) as \( \lambda \to \infty \) for any \( i \in \mathbb{Z}_+ \).

On the other hand, observe that the difference equation (29) is the Euler discretization of the PBR dynamics under the price scheme \( \hat{\tau} \). Then, because \( \dot{B} \) is Lipchitz continuous, the standard argument of the finite difference method implies \( |z^{i,h}(\lambda) - z(t; \lambda)| \to 0 \) as \( h \to 0 \) for each \( i \in \mathbb{Z}_+ \) and \( \lambda \geq 0 \) where \( t_i = ih \) and \( z(t; \lambda) \) is the trajectory of the PBR dynamics under the price scheme \( \hat{\tau} \) with \( z(0; \lambda) = z^*(\lambda) \).23

Because the PBR dynamics is assumed to be convergent, there exists \( \hat{z}(\lambda) \in Z \) such that \( z(t; \lambda) \to \hat{z}(\lambda) \) as \( t \to \infty \) for each \( \lambda \geq 0 \).

Let \( \hat{z}^h(\lambda) \) be an accumulation point of \{\( z^{i,h}(\lambda) \}_{i \geq 0} \}. Take a double sequence \( \{\lambda_m, h_n\}_{m,n} \) such that \( \lambda_m \to \infty \) as \( m \to \infty \) and \( h_n \to 0 \) as \( n \to \infty \). Then, we de-

\[ ^{23}\text{See, for example, Section 6.3.3 of LeVeque (2007).} \]
fine \(s(m, n) = \hat{z}^{k_m}(\lambda_m)\). Because \(\{s(m, n)\}_{m,n}\) is bounded, we can take a convergent subsequence \(\{s(m_k, n_r)\}_{k,r}\). The preceding arguments then imply that, for any \(r\), 
\[
\lim_{k \to \infty} s(m_k, n_r) = z^* \quad \text{whereas, for any } k, \lim_{r \to \infty} s(m_k, n_r) = \hat{z}(\lambda_m).
\]
Therefore,
\[
\lim_{k \to \infty} \hat{z}(\lambda_m) = \lim_{k \to \infty} \lim_{r \to \infty} s(m_k, n_r) = \lim_{r \to \infty} \lim_{k \to \infty} s(m_k, n_r) = z^*.
\] (33)

Because this is true for any convergent subsequence of \(\{\lambda_m, h_n\}_{m,n}\), 
\[
\lim_{m \to \infty} \hat{z}(\lambda_m) = z^*.
\]

Note that \(\hat{z}(\lambda)\) is a perturbed equilibrium of \(\hat{G}_\lambda(x,\theta)\) that is induced by the price scheme \(\hat{\tau}\). Thus, we conclude the result. \(\Box\)

**Proof of Lemma 2.** Suppose \(\bar{x} \in X\) is the single-point global attractor of the APBR dynamics. We claim that \(z'_k(t)\), which is the trajectory of the PBR dynamics, converges to \(\hat{B}_\lambda^k(\bar{x})\). Suppose, to the contrary, \(|z'_k(t) - \hat{B}_\lambda^k(\bar{x})| \geq \varepsilon\) for all \(t\) for some \(\varepsilon > 0\). For such \(\varepsilon\), there is \(T \geq 0\) such that \(|\hat{B}_\lambda^k(x(z(t))) - \hat{B}_\lambda^k(\bar{x})| < \varepsilon\) for all \(t \geq T\). Thus,
\[
\varepsilon \leq |z'_k(t) - \hat{B}_\lambda^k(\bar{x})| \\
\leq |z'_k(t) - \hat{B}_\lambda^k(x(z(t)))| + |\hat{B}_\lambda^k(x(z(t))) - \hat{B}_\lambda^k(\bar{x})| \\
< |z'_k(t) - \hat{B}_\lambda^k(x(z(t)))| + \varepsilon \\
\Rightarrow |z'_k(t) - \hat{B}_\lambda^k(x(z(t)))| > 0
\]
for all \(t \geq T\). Then, because \(\hat{z}'_k(t) = \hat{B}_\lambda^k(x(z(t))) - z'_k(t)\) and \(z'_k(t)\) is continuous, either \(\hat{z}'_k(t) > 0\) for all \(t \geq T\) or \(\hat{z}'_k(t) < 0\) for all \(t \geq T\). However, because \(|z'_k(t) - \hat{B}_\lambda^k(\bar{x})| \geq \varepsilon\) for all \(t\), \(\lim_{t \to \infty} \hat{z}'_k(t) \neq 0\). Therefore, we reach a contradiction in either case because \(z'_k(t)\) is bounded. Because the initial point of \(z'_k(t)\) is arbitrary, we then conclude that \(\hat{B}(\bar{x})\) is the single-point global attractor of the PBR dynamics. \(\Box\)
References


